

# Wave dispersion in a damped beam supported by cubic nonlinear springs: A multiscale freewave approach

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Researchers are drawn to exploring wave dispersion in nonlinear systems because of the amplitude-dependent tunability of the bandgap. This paper investigates the amplitude-dependent wave dispersion in continuous beam structures supported periodically by nonlinear springs. Additionally, it examines the influence of inherent beam damping on wave dispersion. The analytical framework consists of homogenization of the unit cell and the method of multiple scales with two distinct time scales to derive the wave dispersion equation. The proposed analytical approach for nonlinear wave propagation is validated through numerical finite element simulations. It is observed that the frequency shift is positive for hardening and negative for softening supports. Following this, the dispersion shift over time in the damped systems is examined by considering viscous and strain rate-dependent damping. The sensitivity of strain rate damping to propagation constant and the independence of viscous damping from propagation constant are thoroughly investigated. In a damped system, the frequency shift diminishes over time as the amplitude decreases reducing the effect of nonlinearity. This study opens up avenues for controlling or filtering vibrations through the tunable bandgap of continuous nonlinear metamaterials.

Keywords: Homogenisation, Method of multiple scales, Frequency shift, Amplitude modulation, Amplitude dependant dispersion

## I. INTRODUCTION

Over the past decade, wave dispersion in linear metamaterials has garnered considerable attention from researchers due to its complex dispersion patterns, filtering capabilities, and the emergence of frequency bandgaps. The presence of nonlinearities within these systems has further piqued the interest of the engineering community, revealing fascinating phenomena such as amplitude-dependent dispersion and alterations in group velocity with weak nonlinear effects, as well as the propagation of solitary waves like solitons under strong nonlinear conditions. This exploration of wave propagation in nonlinear metamaterial has led to the proposal of various engineering applications, including diodes [1, 2], switches [3], and filters [4].

Several recent studies have analyzed amplitude-dependent dispersion in discrete nondispersive systems such as monoatomic and diatomic chains with cubic nonlinearity using perturbation method [5], nonlinear triatomic metamaterial [6, 7], monoatomic chain with nonlinear embedded resonator [8], nonlinear monoatomic chain with embedded resonator [9, 42], nonlinear ro-

ton like chain [11] etc. Additionally, nonlinear behavior such as amplitude-induced bandgap [10, 12], self-switching functionality [13], bridging coupling mechanism [14], band tunability [43] and nonreciprocity [44, 45] etc has been observed in variety of discrete systems. Different analytical methods such as Lindstedt–Poincaré perturbation technique, method of multiscale, harmonic balance method, method of averaging, Jacobi’s elliptic balance method [60] etc [15, 16] have been used in deriving analytical solutions of the nonlinear dynamic systems in the past. However, the method of multiple scales is well suited for the damped system as it solves for amplitude modulation [17–20]. In the case of nonlinear continuous systems, researchers generally tackle finite systems and obtain frequency response function [21–28]. In nonlinear continuum elastic media with topological mechanics the method of multiple scales is widely used [58, 59]. In the case of continuous infinite structures, the nonlinear dispersion shift is obtained for a few systems such as a bar with periodically embedded resonators [29–31], beam with periodic resonators [32, 33] etc. The quasi-static wide bandgap can be induced by a beam on an elastic foundation for linear systems [35]. Additionally by incorporating nonlinear supports, the tunable quasi-static bandgap can be obtained [36]. This has motivated us to study the amplitude-dependent wave dispersion in infinite long periodically supported damped beams.

An infinite beam with damping and nonlinear supports has numerous practical applications. For instance,

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77 railway tracks can be modeled as infinitely long Euler-133  
 78 Bernoulli beams resting on nonlinear spring-dashpot sys-  
 79 tems, making the study of flexural wave propagation es-134  
 80 sential for the analysis, design, and health monitoring of135  
 81 railway infrastructure [46–48]. Additionally, functionally  
 82 graded foams, which can be modeled as beams supported136  
 83 by nonlinear springs with damping [55], are widely used  
 84 in various fields. These include acoustic control [53],  
 85 shock and energy absorption [49, 50], as well as appli-137  
 86 cations in the automotive industry [51] and biomedical  
 87 instruments [52]. Furthermore, beams on periodic elastic138  
 88 foundations are key for achieving wider bandgap regions139  
 89 and stronger attenuation capabilities [56], enhancing pas-140  
 90 sive vibration control [54] and other advanced engineer-141  
 91 ing solutions.142

92 Historically, researchers have directed their attention143  
 93 towards both nonlinear and linear wave propagation144  
 94 within discrete metamaterials, while investigations into145  
 95 wave propagation in continuous metamaterials have pri-146  
 96 marily focused on linear systems. Concerning nonlinear  
 97 continuous systems, researchers have primarily explored147  
 98 frequency response functions within various finite sys-  
 99 tems, leaving a limited examination of wave propagation148  
 100 within continuous beam systems incorporating nonlinear-  
 101 ity [34].149

102 To address this gap, the study delves into the149  
 103 amplitude-dependent, time-varying dispersion relation of150  
 104 a damped beam supported by nonlinear springs. Specifi-  
 105 cally, it explores the wave dispersion relation of a damped  
 106 beam supported by periodic nonlinear structures, con-  
 107 sidering two types of damping—viscous and strain rate151  
 108 damping—alongside both softening and hardening cubic152  
 109 nonlinearity. Employing a multiscale method with two  
 110 distinct time scales, the study analytically derives the  
 111 wave dispersion equation. The accuracy of the analytical153  
 112 solution is confirmed through comparison with numerical  
 113 results. Initially, the paper discusses the frequency shift154  
 114 in undamped systems affected by hardening and soften-  
 115 ing nonlinearities. Subsequently, it examines the disper-  
 116 sion shift over time in damped systems, distinguishing156  
 117 between viscous and strain rate damping. The sensitiv-  
 118 ity of strain rate damping to propagation constants and158  
 119 the independence of viscous damping from propagation  
 120 constants are thoroughly investigated. In summary, the  
 121 paper extensively explores the amplitude-dependent non-  
 122 linear dispersion in infinitely long damped beams period-  
 123 ically supported by nonlinear springs. These are vital for  
 124 modeling railway tracks, functionally graded foams, and  
 125 enhancing passive vibration control in various engineer-162  
 126 ing applications.163

## 127 II. METHODOLOGY

128 Considering the flexural wave propagation in infinitely169  
 129 long Euler Bernoulli beam supported by cubic nonlinear  
 130 springs as depicted in Fig. 1(a), the following analytical170  
 132 formulation is derived.

### A. Governing equation

The governing equation of motion of the representative unit cell shown in Fig. 1(b) can be written as [37]:

$$\begin{aligned} & \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 w}{\partial x^2} + C_s \frac{\partial^3 w}{\partial x^2 \partial t} \right) + \rho A \frac{\partial^2 w}{\partial t^2} \\ & + C_a \frac{\partial w}{\partial t} + \left( \tilde{k}_1 w + \tilde{k}_3 w^3 \right) \delta(x) = 0 \end{aligned} \quad (1)$$

where,  $EI$  = flexural rigidity,  $C_s$  = strain rate-dependent damping,  $C_a$  = velocity-dependent viscous damping,  $\rho$  = density,  $A$  = cross-section area,  $\tilde{k}_1$  and  $\tilde{k}_3$  are liner and nonlinear spring stiffness,  $w$  denotes transverse deflection,  $x$  = distance, and  $t$  represents time. Further, the Dirac delta function was approximated using a homogenization approach in which the stiffness of the spring was scaled down by the inverse of the length of the unit cell ( $1/l$ ) as [38].

$$\begin{aligned} & \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 w}{\partial x^2} + C_s \frac{\partial^3 w}{\partial x^2 \partial t} \right) + \rho A \frac{\partial^2 w}{\partial t^2} \\ & + C_a \frac{\partial w}{\partial t} + \frac{1}{l} \left( \tilde{k}_1 w + \tilde{k}_3 w^3 \right) = 0 \end{aligned} \quad (2)$$

Further, the governing equation can be written as

$$\frac{\partial^2 w}{\partial t^2} + \omega_n^2 \frac{\partial^4 w}{\partial x^4} + k_1 w + c_1 \frac{\partial w}{\partial t} + c_2 \frac{\partial^5 w}{\partial x^4 \partial t} + k_3 w^3 = 0 \quad (3)$$

where,  $\omega_n^2 = \frac{EI}{\rho A}$ ,  $c_1 = \frac{C_s}{\rho A}$ ,  $c_2 = \frac{C_a}{\rho A}$ ,  $k_1 = \frac{1}{\rho A l} \tilde{k}_1$  and  $k_3 = \frac{1}{\rho A l} \tilde{k}_3$

### B. Multiple scales Method

The nonlinear governing equation given in Eq. (3) has been solved using the method of multiple scales by introducing the scaling parameter  $\epsilon$  with damping and nonlinear terms as:

$$\frac{\partial^2 w}{\partial t^2} + \omega_n^2 \frac{\partial^4 w}{\partial x^4} + k_1 w + \epsilon \left( c_1 \frac{\partial w}{\partial t} + c_2 \frac{\partial^5 w}{\partial x^4 \partial t} + k_3 w^3 \right) = 0 \quad (4)$$

The nonlinear partial differential equation, as presented in Eq. (4), involves independent variables  $x$  and  $t$  corresponding to spatial and temporal dimensions, respectively. The method of multiple scales can be applied by employing slow time scales or large space scales. Here, by employing a multiscale approach, additional scales are introduced to account for slow time scale, denoted as  $T_1 = \epsilon t$ . These augment the original scales for time  $T_0 = t$ . Another way of the multiple scales method by employing spatial expansion is demonstrated in Appendix A. The time derivatives can be written as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \epsilon \frac{\partial}{\partial T_1} \quad (5)$$

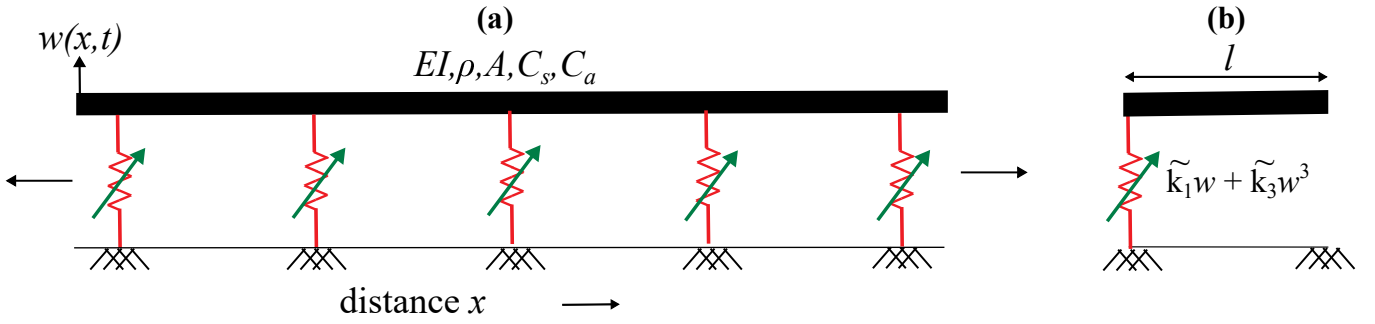


FIG. 1. (a) The Euler Bernoulli beam periodically supported by nonlinear springs (b) The representative unit cell

Further, the solution can be obtained as

$$w(x, t, \epsilon) = \sum_{n=0}^1 \epsilon^n w_n(x, T_0, T_1) + O(\epsilon^2) \quad (6)$$

By substituting the solution given in Eq. (6) into the partial differential equation (Eq. (4)) and employing the temporal derivatives provided in Eq. (5), yields

$$\begin{aligned} & \epsilon^2 \left( \frac{\partial^2 w_0}{\partial T_1^2} + 2 \frac{\partial^2 w_1}{\partial T_1 \partial T_0} \right) + \epsilon^3 \frac{\partial^2 w_1}{\partial T_1^2} \\ & + \epsilon \left( 2 \frac{\partial^2 w_0}{\partial T_1 \partial T_0} + \frac{\partial^2 w_1}{\partial T_0^2} \right) + \frac{\partial^2 w_0}{\partial T_0^2} + \omega_n^2 \left( \epsilon \frac{\partial^4 w_1}{\partial x^4} + \frac{\partial^4 w_0}{\partial x^4} \right) \\ & + k_1 (w_0 + \epsilon w_1) + c_1 \epsilon \left( \frac{\partial w_0}{\partial T_0} + \epsilon^2 \frac{\partial w_1}{\partial T_1} + \epsilon \left( \frac{\partial w_1}{\partial T_0} + \frac{\partial w_0}{\partial T_1} \right) \right) \\ & + c_2 \epsilon \left( \epsilon^2 \frac{\partial^5 w_1}{\partial x^4 \partial T_1} + \epsilon \left( \frac{\partial^5 w_1}{\partial x^4 \partial T_0} + \frac{\partial^5 w_0}{\partial x^4 \partial T_1} \right) + \frac{\partial^5 w_0}{\partial x^4 \partial T_0} \right) \\ & + \epsilon k_3 (w_0 + \epsilon w_1)^3 = 0 \end{aligned} \quad (7)$$

Further, collecting terms with a similar power of  $\epsilon$ , the following equation can be written.

$$\begin{aligned} & \frac{\partial^2 w_0}{\partial T_0^2} + \frac{\partial^4 w_0}{\partial x^4} + k_1 w_0 \\ & + \epsilon \left( \frac{\partial^2 w_1}{\partial T_0^2} + 2 \frac{\partial^2 w_0}{\partial T_1 \partial T_0} + \frac{\partial^4 w_1}{\partial x^4} + k_1 w_1 \right) \\ & + \epsilon \left( c_1 \frac{\partial w_0}{\partial T_0} + c_2 \frac{\partial^5 w_0}{\partial x^4 \partial T_0} + k_3 w_0^3 \right) + O(\epsilon^2) = 0 \end{aligned} \quad (8)$$

The governing partial differential equations can be expressed through the consolidation of terms with similar powers of  $\epsilon$  as.

$$R_0 + R_1 \epsilon + O(\epsilon^2) = 0 \quad (9)$$

where

$$R_0 : \frac{\partial^2 w_0}{\partial T_0^2} + \frac{\partial^4 w_0}{\partial x^4} + k_1 w_0 = 0 \quad (10)$$

$$\begin{aligned} R_1 : & \frac{\partial^2 w_1}{\partial T_0^2} + 2 \frac{\partial^2 w_0}{\partial T_1 \partial T_0} + \frac{\partial^4 w_1}{\partial x^4} + k_1 w_1 \\ & + c_1 \frac{\partial w_0}{\partial T_0} + c_2 \frac{\partial^5 w_0}{\partial x^4 \partial T_0} + k_3 w_0^3 = 0 \end{aligned} \quad (11)$$

Eq. (9) expands the nonlinear equation of motion in orders of  $(\epsilon)$ , matching it to the zeroth order and first order in  $(\epsilon)$ . The solutions  $w_0$ , and  $w_1$  can be obtained by solving equations  $R_0 = 0$  and  $R_1 = 0$  in progression.

In the case of flexural wave solution in infinite structure, the boundary complexities can be ignored and the plane wave solution can be assumed for the equation  $R_0 = 0$  as [39].

$$w_0 = A(T_1) e^{i(\kappa x - \omega T_0)} + \bar{A}(T_1) e^{-i(\kappa x - \omega T_0)} \quad (12)$$

Substituting Eq. (12) in Eq. (10) the dispersion relation for the linear system can be derived as

$$\kappa^4 - \omega^2 + k_1 = 0 \quad (13)$$

Further, substituting Eq. (12) in Eq. (11), the following equation can be obtained

$$\begin{aligned} & \frac{\partial^2 w_1}{\partial T_0^2} + \frac{\partial^4 w_1}{\partial x^4} + k_1 w_1 \\ & = \left( 2i\omega \frac{\partial A}{\partial T_1} + ic_1 \omega A + ic_2 \kappa^4 \omega A - 3k_3 A^2 \bar{A} \right) e^{i(\kappa x - \omega T_0)} \\ & - k_3 A^3 e^{i(3\kappa x - 3\omega T_0)} + cc \end{aligned} \quad (14)$$

The particular solution of Eq. (14) contains secular terms which lead to nonuniform expansion in scaled time. Since the linear operator  $\left( \frac{\partial^2}{\partial T_0^2} + \frac{\partial^4}{\partial x^4} + k_1 \right)$  is self-adjoint, as demonstrated in the Appendix B, the solvability condition for eliminating secular terms can be derived. This condition is obtained by equating the forcing terms responsible for generating the secular terms to zero, which leads to the following partial differential equation governing the amplitude  $A$ .

$$\begin{aligned} & 2i\omega \frac{\partial A}{\partial T_1} + ic_1 \omega A + ic_2 \kappa^4 \omega A - 3k_3 A^2 \bar{A} = 0 \\ & i \frac{\partial A}{\partial T_1} = -iA \left( \frac{c_1 + c_2 \kappa^4}{2} \right) + \frac{3}{2} k_3 \omega^{-1} A^2 \bar{A} \end{aligned} \quad (15)$$

The solution of amplitude  $A$  in polar form can be assumed as  $A = \frac{1}{2} a(T_1) e^{i\beta(T_1)}$  and its complex conjugate,  $\bar{A} = \frac{1}{2} a(T_1) e^{-i\beta(T_1)}$ . Further, the frequency shift due to

nonlinearity will be later demonstrated and reflected by the complex-valued phase of amplitude modulation. As the phase  $\beta$  is a function of slow time scale ( $T_1$ ) it essentially captures the frequency shift due to nonlinearity. To determine the phase, by substituting the amplitude  $A$  and  $\bar{A}$  into Eq. (15) the following equations can be obtained.

$$\begin{aligned} & i\frac{1}{2}\left(e^{i\beta}\frac{\partial a}{\partial T_1} + ie^{i\beta}a\frac{\partial \beta}{\partial T_1}\right) \\ & = -i\left(\frac{c_1 + c_2\kappa^4}{2}\right)\frac{a}{2}e^{i\beta} + \frac{3}{2}k_3\omega^{-1}e^{i\beta}\frac{a^3}{8} \end{aligned} \quad (16)$$

Further separating the real and imaginary parts following equations can be obtained.

$$\frac{\partial a}{\partial T_1} = -\left(\frac{c_1 + c_2\kappa^4}{2}\right)a \quad (17)$$

$$\frac{\partial \beta}{\partial T_1} = -\frac{3}{8}k_3\omega^{-1}a^2 \quad (18)$$

The amplitude modulation can be obtained by solving Eq. (17) as

$$a = a_0 e^{-\left(\frac{c_1 + c_2\kappa^4}{2}\right)T_1} \quad (19)$$

Further, the frequency shift can be obtained by substituting Eq. (19) into Eq. (18) as follows.

$$\begin{aligned} & \frac{\partial \beta}{\partial T_1} = -\frac{3}{8}k_3\omega^{-1}a_0^2 e^{-(c_1 + c_2\kappa^4)T_1} \\ & \beta = \frac{3}{8}\frac{k_3\omega^{-1}a_0^2}{(c_1 + c_2\kappa^4)}e^{-(c_1 + c_2\kappa^4)T_1} + \beta_0 \end{aligned} \quad (20)$$

Further, the initial condition has been assumed as at  $T_1 = 0, \beta = 0$  and so,  $\beta_0 = -\frac{3}{8}\frac{k_3\omega^{-1}a_0^2}{(c_1 + c_2\kappa^4)}$ . The shift in frequency can be determined as

$$\begin{aligned} & \beta = \frac{3}{8}\frac{k_3\omega^{-1}a_0^2}{(c_1 + c_2\kappa^4)}e^{-(c_1 + c_2\kappa^4)T_1} - \frac{3}{8}\frac{k_3\omega^{-1}a_0^2}{(c_1 + c_2\kappa^4)} \\ & \beta = -\frac{3}{8}\frac{k_3\omega^{-1}a_0^2}{(c_1 + c_2\kappa^4)}\left(1 - e^{-(c_1 + c_2\kappa^4)T_1}\right) \end{aligned} \quad (21)$$

The wave solution can be written by substituting amplitude Eq. (19) and frequency shift Eq. (21) into Eq. (12) as

$$\begin{aligned} & w_0 = \frac{a_0}{2} e^{-\left(\frac{c_1 + c_2\kappa^4}{2}\right)T_1} \\ & e^{-i\frac{3}{8}\frac{k_3\omega^{-1}a_0^2}{(c_1 + c_2\kappa^4)}T_1}\left(1 - e^{-(c_1 + c_2\kappa^4)T_1}\right)T_1 e^{i(\kappa x - \omega T_0)} + cc \end{aligned} \quad (22)$$

Further, substituting  $T_0 = t$  and  $T_1 = \epsilon t$ ,

$$\begin{aligned} & w_0 = \frac{a_0}{2} e^{-\left(\frac{c_1 + c_2\kappa^4}{2}\right)\epsilon t} \\ & e^{-i\frac{3}{8}\frac{k_3\omega^{-1}a_0^2}{(c_1 + c_2\kappa^4)}\epsilon t}\left(1 - e^{-(c_1 + c_2\kappa^4)\epsilon t}\right)\epsilon t e^{i(\kappa x - \omega t)} + cc \end{aligned} \quad (23)$$

The frequency shift can be written as

$$\omega_s = \omega + \frac{3}{8}\frac{k_3\omega^{-1}a_0^2}{(c_1 + c_2\kappa^4)}t\left(1 - e^{-(c_1 + c_2\kappa^4)\epsilon t}\right) \quad (24)$$

To further clarify the frequency shift derived in Eq. (24), it is important to highlight that the system's nonlinearity leads to an amplitude-dependent frequency shift. Initially, when  $t$  is small and the amplitude is large, the nonlinearity induces a substantial frequency shift. As time progresses and damping reduces the amplitude, the corresponding frequency shift diminishes. This behavior illustrates the direct correlation between the wave amplitude and the magnitude of the frequency shift in nonlinear systems. However, in the case of an undamped system, substituting  $c_1 = c_2 = 0$  in Eq. (15) following frequency shift and wave solution can be obtained.

$$\omega_s = \omega + \frac{3k_3a_0^2}{8\omega}\epsilon \quad (25)$$

$$w_0 = a_0 e^{-\left(\frac{3k_3a_0^2}{8\omega}\right)\epsilon t} e^{i(\kappa x - \omega t)} + cc \quad (26)$$

The amplitude-dependant dispersion relation of an undamped beam supported by nonlinear springs can be obtained by squaring Eq. (25), substituting  $\kappa^4 + k_1$  at  $\omega^2$  and neglecting terms of  $\epsilon^2$  as

$$\omega_s^2 = \kappa^4 + k_1 + \epsilon\frac{3k_3a_0^2}{4} \quad (27)$$

Note that, the strength of cubic nonlinearity is generally considered by factor  $\Pi = \frac{\epsilon k_3 a_0^2}{k_1}$ . The method of multiple scales is applicable in the conservative bounds as  $|\Pi| < 0.1$  [41]. Moreover, when the system is damped, the amplitude is always decreasing and is independent of the strength of cubic nonlinearity as time progresses which makes the system always stable [19].

### C. Finite element modelling

A finite element formulation of the Euler-Bernoulli beam on nonlinear elastic springs has been developed for numerical validation. A governing strong form equation for a representative unit cell is derived as Eq. (4). Let,  $w(x, t) = \psi(x)q(t)$ , where,  $q(t) = \{u_i, u'_i, u_j, u'_j\}^T$ ;  $u_i$  and  $u'_i$  are displacement and slope at  $i^{th}$  node at time  $t$ . Further  $\psi(x)$  is shape function defined by Hermite polynomial as  $\psi(x) = [\psi_1 \ \psi_2 \ \psi_3 \ \psi_4]$  where,

$$\begin{aligned} \psi_1(x) &= \frac{1}{l^3} (2x^3 - 3x^2l + l^3) \\ \psi_2(x) &= \frac{1}{l^3} (x^3l - 2x^2l^2 + xl^3) \\ \psi_3(x) &= \frac{1}{l^3} (-2x^3 + 3x^2l) \\ \psi_4(x) &= \frac{1}{l^3} (x^3l - x^2l^2) \end{aligned} \quad (28)$$



Employing the Galerkin method, the weak form of a strong form equation given in Eq. (4) can be derived as [40]

$$\begin{aligned}
 & \underbrace{\left[ \int_0^l \psi^T \psi dx \right]}_{\mathbf{M}} \ddot{q}(t) + \underbrace{\left[ \frac{\omega_n^2}{l^3} \int_0^l \frac{d^2 \psi}{dx^2}^T \frac{d^2 \psi}{dx^2} dx \right]}_{\mathbf{K}} q(t) + \underbrace{[k_1 \psi(0)]}_{\mathbf{L}} q(t) \\
 & + \underbrace{\left[ \epsilon c_1 \int_0^l \psi^T \psi dx + \epsilon c_2 \int_0^l \frac{d^2 \psi}{dx^2}^T \frac{d^2 \psi}{dx^2} dx \right]}_{\mathbf{C}} \dot{q}(t) + \underbrace{[\epsilon k_3 \psi(0)^3]}_{\mathbf{N}} q(t)^3 = 0
 \end{aligned} \tag{29}$$

The typical element mass matrix ( $\mathbf{M}$ ), stiffness matrix ( $\mathbf{K}$ ), mass and stiffness proportional Rayleigh damping matrix ( $\mathbf{C}$ ) [57], and the linear and nonlinear spring stiffness matrices ( $\mathbf{L}$  and  $\mathbf{N}$ ) for a single unit have been derived as follows.

$$\mathbf{M} = \frac{1}{420} \begin{pmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{pmatrix}$$

$$\mathbf{K} = \omega_n^2 \begin{pmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{pmatrix}$$

$$\begin{aligned}
 \mathbf{C} = & \frac{c_1}{420} \begin{pmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{pmatrix} \\
 & + \frac{c_2}{l^3} \begin{pmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{pmatrix}
 \end{aligned}$$

$$\mathbf{L} = \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{N} = \begin{pmatrix} k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Further, the global matrices ( $\mathbb{J}_g$ ) have been obtained and the global equation of motion can be written as follows.

$$\mathbf{M}_g \ddot{\mathbf{v}} + \mathbf{K}_g \mathbf{v} + \mathbf{L}_g \mathbf{v} + \epsilon (\mathbf{C}_g \dot{\mathbf{v}} + \mathbf{N}_g \mathbf{v}^3) = 0 \tag{33}$$

where  $\mathbf{v}$  is the global displacement vector.

### III. RESULTS AND DISCUSSION

In this section, the proposed theory for nonlinear wave propagation has been validated with the numerical solution. Further, this section discusses the wave dispersion

phenomena within both undamped and damped Euler-Bernoulli beams supported by cubic nonlinearities, exploring both hardening and softening nonlinear behaviors. Additionally, it thoroughly examines the impact of viscous damping and strain rate damping on damped beams.

#### A. Numerical validation

The nonlinear dispersion relation can be derived using two approaches. The first is the free wave approach, where the propagation constant ( $\kappa$ ) is fixed by applying harmonic initial condition, and the corresponding frequency ( $\omega$ ) is determined. Alternatively, the driven wave approach fixes the frequency ( $\omega$ ) through harmonic boundary conditions, leading to the determination of the propagation constant. However, in dispersive media, the boundary complexity decreases the amplitude of harmonic boundary conditions in the far field, impacting the amplitude-dependent dispersion. Consequently, for the nonlinear dispersion relation of infinite continuous beams, the free wave approach is preferable for numerical validation.

The MATLAB ODE45 function has been used to obtain the solution of the Eq. (33). For this, a very long beam has been simulated for a long time to get accurate fast Fourier transform (FFT) results but not so long time that the boundary reflections affect the solution. The FFT is calculated in both space and time to get wavenumber ( $\kappa$ ) as well as frequency ( $\omega$ ). According to the free-wave approach, the frequency shift is obtained by fixing the wavenumber by imposing the initial condition as  $w(x, 0) = A \cos(\kappa x)$ ,  $w'(x, 0) = -A\kappa \sin(\kappa x)$ .

For illustration purposes, the wave number  $\kappa = 0.5$  and the amplitude  $A = 1$  are applied in the initial condition to the beam shown in Fig. 2(a). Furthermore, the stiffness of the elastic springs is taken as  $k_1 = 1$ ,  $k_3 = -1$  and  $\epsilon = 1$  for the undamped system. In Fig. 2(b), the displacement profile of the center portion of the beam

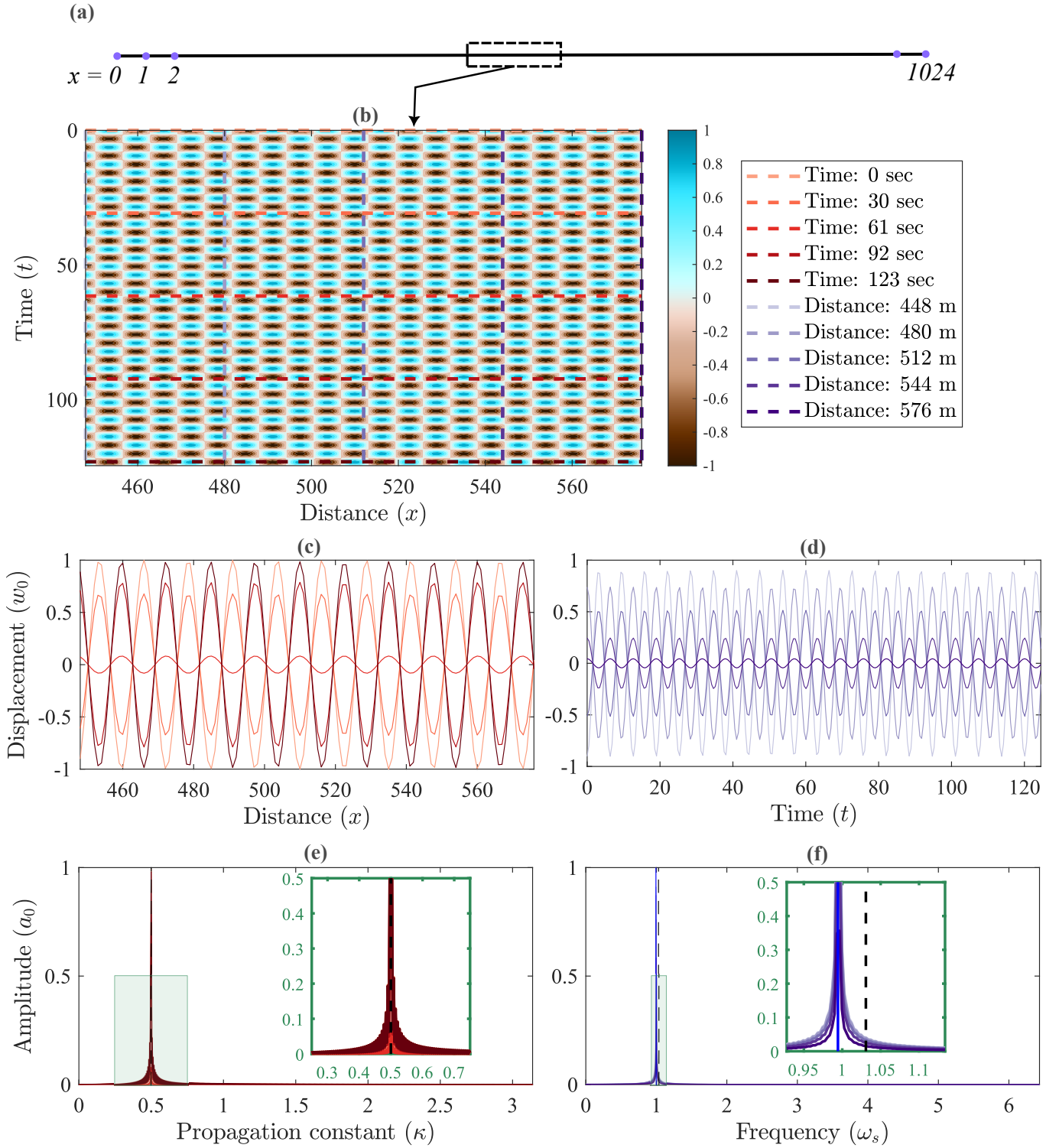


FIG. 2. (a) The long Euler Bernoulli beam with  $c_1 = c_2 = 0, k_3 = -1, k_1 = 1, A = 1, \epsilon = 0.1$ , (b) Displacement contour of the middle portion of the beam shown by the dashed box, (c) Displacement versus distance plot at different instances of time shown in horizontal dashed lines with orange shades in (b), (d) Displacement versus time plot at different locations shown in vertical dashed lines with violet shades in (b), (e) FFT plots corresponding to displacement profiles shown in (c), (f) FFT plots corresponding to time histories shown in (d).

with respect to space and time is demonstrated. Further, the beam displacement with respect to space at different times as shown by the horizontal dashed line in Fig. 2(b) are plotted in Fig. 2(c), and its FFT plots

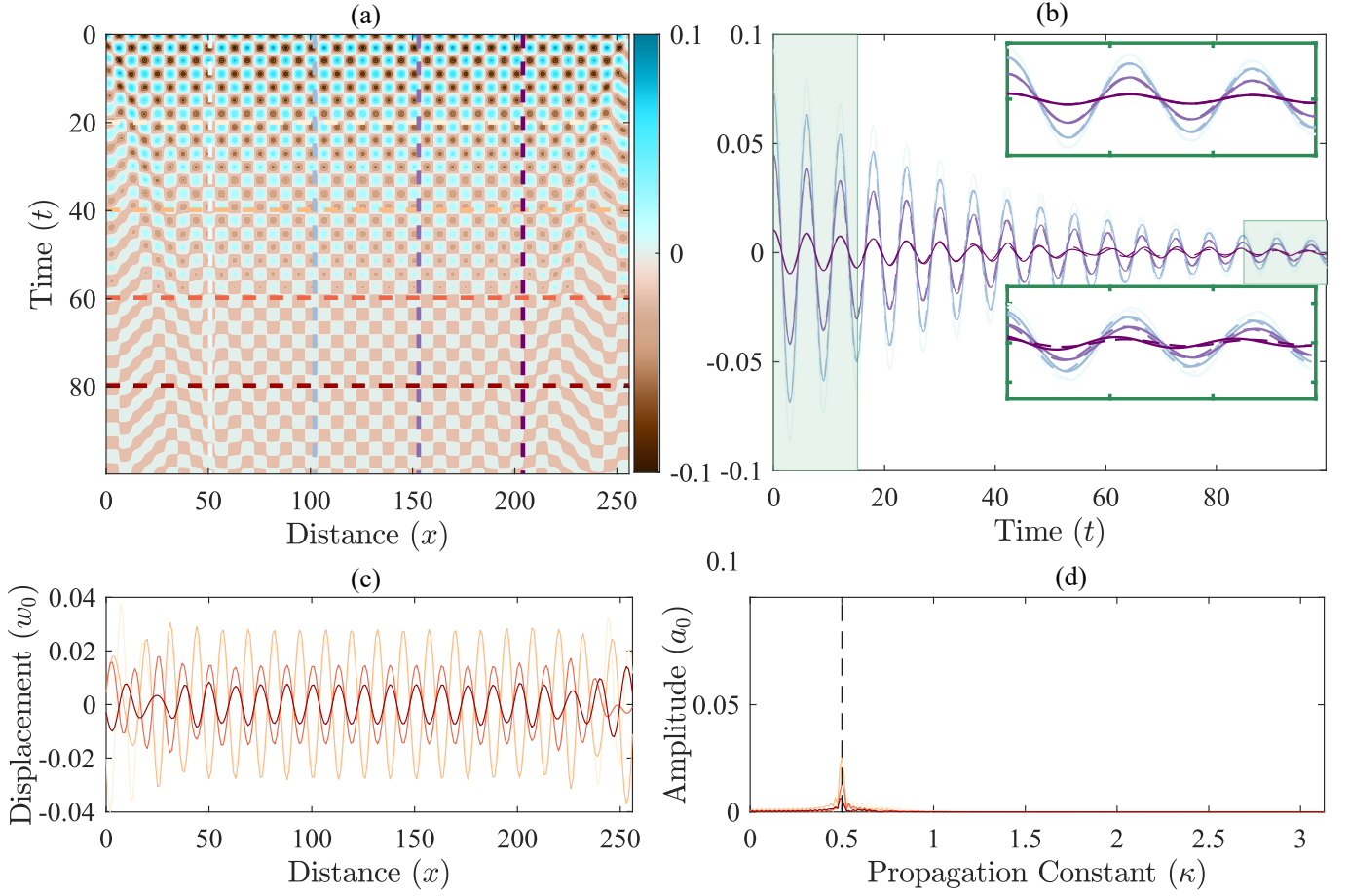


FIG. 3. (a) Displacement contour of the Euler Bernoulli beam with  $c_1 = c_2 = 0.05, k_3 = 1, k_1 = 1, A = 0.1, \epsilon = 1$ ; (b) Displacement versus time plot at different locations shown in vertical dashed lines with violet shades in (a) where the solid lines depict numerical solution and the dashed lines depict analytical solution, moreover the magnified plots are shown inside to showcase the better validation in the initial time window and degraded validation in later time window; (c) Displacement versus distance plot at different instances of time shown in horizontal dashed lines with orange shades in (a); (d) FFT plots corresponding to displacement profiles shown in (c).

are plotted in Fig. 2(e) which shows peak amplitude at the wavenumber ( $\kappa = 0.5$ ) applied in the initial condition. Furthermore, the beam displacement with respect to time at different spaces located by vertical dashed lines as shown in Fig. 2(b) are plotted in Fig. 2(d), and its FFT can be observed in Fig. 2(f). The peak amplitude frequency in Fig. 2(f) matches exactly with the analytically obtained nonlinear frequency shown with a solid blue line. As the nonlinearity is soft the frequency shift is negative which can be seen in the magnified plot in Fig. 2(f), where the frequency of the linear system is shown by a black dashed line. This validates the proposed theory of the free wave approach for an undamped system.

The damped system has been validated further in Fig. 3 with the following parameters: spring stiffness with hard nonlinearity as  $k_1 = 1, k_3 = 1$ , the damping factor  $c_1 = c_2 = 1$  and  $\epsilon = 1$ . In addition, the initial condition is kept similar to the undamped case as the wave number  $\kappa = 0.5$  and the amplitude  $A = 1$ . The displacement profile of the beam in space-time is illustrated in Fig.

3(a) which shows the decreases in amplitude as time progresses. However, the wavelength is constant as shown in Fig. 3(c) in which the displacement is plotted at different times shown by horizontal dashed lines in Fig. 3(a). Further, its FFT is plotted in Fig. 3(d), which shows the consistent wavenumber at different times. Furthermore, the time histories at different locations shown by vertical dashed lines in Fig. 3(a) are plotted in Fig. 3(b). The decrease in amplitude due to damping reduces the frequency shift with time, therefore instead of FFT of time histories for validation, the analytical displacement time histories are plotted with dashed lines to validate the proposed theory for the damped system. In magnified subplots of Fig. 3(b), the analytical and numerical solution matches exactly in the initial phase (Fig. 3(b1)) however, they deviate in long time as shown in Fig. 3(b2) as the analytical solution is valid till time inversely proportional to the scaling factor ( $\epsilon$ ) [16].

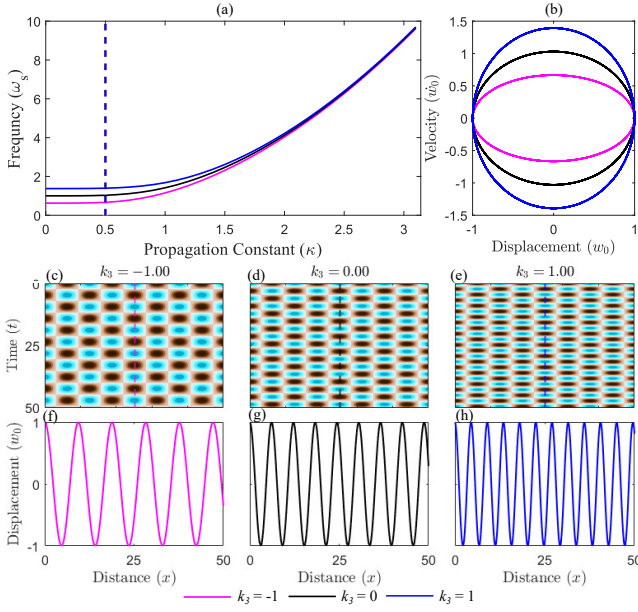


FIG. 4. (a) Dispersion relation plot for undamped system with  $c_1 = c_2 = 0$ ,  $k_1 = 1$  and nonlinear stiffness parameter  $k_3 = -1, 0$  and  $1$ ; (b) Phase portrait for propagation constant  $\kappa = 0.5$ ; (c) displacement profile for  $\kappa = 0.5$ ,  $k_3 = -1$ ; (d) displacement profile for  $\kappa = 0.5$ ,  $k_3 = 0$ ; (e) displacement profile for  $\kappa = 0.5$ ,  $k_3 = 1$ ; (f) Displacement time history at middle node of the beam with  $\kappa = 0.5$ ,  $k_3 = -1$ ; (g) Displacement time history at the middle node of the beam with  $\kappa = 0.5$ ,  $k_3 = 0$ ; (h) Displacement time history at the middle node of the beam with  $\kappa = 0.5$ ,  $k_3 = 1$ .

## B. Undamped system

This section presents an analysis of the dispersion relation of the nonlinear undamped system based on the proposed analytical plane wave solution. Initially, the influence of soft and hard nonlinearity on wave dispersion is investigated, as illustrated in Fig. 4, considering nonlinear spring stiffness values of  $k_3 = -1, 0$ , and  $1$ . An amplitude of  $A = 1$  and a scaling parameter of  $\epsilon = 1$  have been assumed. In Fig. 4(a), the dispersion plot is generated by varying the propagation constant ( $\kappa$ ) from  $0$  to  $\pi$  to obtain the frequency  $\omega_s$  using Eq. (25). As the frequency shift is inversely proportional to the frequency, it can be noticed that frequency shift is higher at low propagation constants and it diminishes at higher wavenumbers. Subsequently, in Fig. 4(c-e), we display the displacement contours for the free vibration of the beam for  $\kappa = 0.5$ . It is noteworthy that while the wave number remains consistent across all three figures, the temporal frequency increases as the nonlinear spring stiffness escalates. Similarly, in Fig. 4(f-h), we plot the response of the midpoint of the beam. Additionally, the corresponding phase portrait is depicted in Fig. 4(b), clearly illustrating the positive and negative shifts in frequency for the hardening and softening systems, respectively, from the linear dispersion.

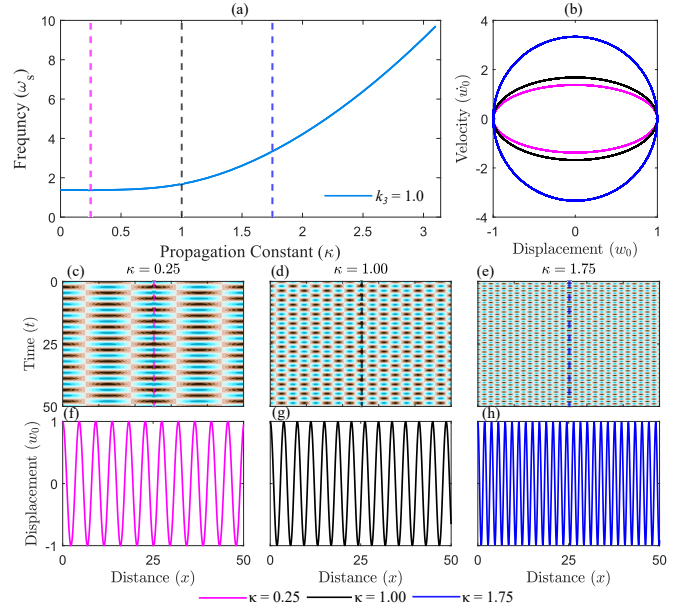


FIG. 5. (a) Dispersion relation plot for the undamped system with  $c_1 = c_2 = 0$ ,  $k_1 = 1$ ,  $k_3 = 1$ ; (b) Phase portrait for propagation constant  $\kappa = 0.25, 1.00$  and  $1.75$ ; (c) displacement profile for  $\kappa = 0.25$ ,  $k_3 = 1$ ; (d) displacement profile for  $\kappa = 1.00$ ,  $k_3 = 1$ ; (e) displacement profile for  $\kappa = 1.75$ ,  $k_3 = 1$ ; (f) Displacement time history at middle node of the beam with  $\kappa = 0.25$ ,  $k_3 = 1$ ; (g) Displacement time history at middle node of the beam with  $\kappa = 1.00$ ,  $k_3 = 1$ ; (h) Displacement time history at middle node of the beam with  $\kappa = 1.75$ ,  $k_3 = 1$ .

Additionally, to analyze the impact of propagation constant, three different propagation constants:  $\kappa = 0.25, 1$ , and  $2$  have been considered, for a nonlinear system with hardening stiffness  $k_3 = 1$ , as depicted in the dispersion plot in Fig. 5(a). Furthermore, the displacement profile of free vibration is illustrated using contour plots in Fig. 5(c-e). The temporal response of the midpoint of the beam is shown in Fig. 5(f-h), accompanied by its corresponding phase portrait in Fig. 5(b). It is evident from these analyses that as the propagation constant increases, the corresponding frequency also increases, with the increment being minimal until propagation constant  $\kappa = 1$ .

## C. Damped system

The effect of viscous damping and strain rate damping along with system nonlinearity on wave dispersion has been discussed in this section. The following three types of systems have been discussed.

1. Viscous damping  $c_1 = 0.25$  and Strain rate damping  $c_2 = 0$
2. Viscous damping  $c_1 = 0$  and Strain rate damping  $c_2 = 0.25$



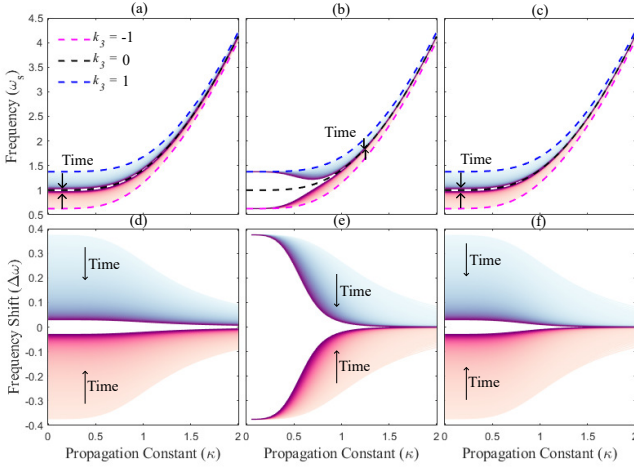


FIG. 6. Dispersion relation plots at different time instances are illustrated for hardening system with blue shades and softening system with shades of pink for (a) only viscous damping  $c_1 = 0.25$  and  $c_2 = 0$ , (b) only strain rate damping  $c_1 = 0$  and  $c_2 = 0.25$  and (c) viscous damping and strain rate damping  $c_1 = 0.25$  and  $c_2 = 0.25$ ; Frequency shift versus propagation constant ( $\kappa$ ) plots at different time instances are illustrated for hardening system with blue shades and softening system with shades of pink for (d) only viscous damping  $c_1 = 0.25$  and  $c_2 = 0$ , (e) only strain rate damping  $c_1 = 0$  and  $c_2 = 0.25$  and (f) viscous damping and strain rate damping  $c_1 = 0.25$  and  $c_2 = 0.25$

### 3. Viscous damping $c_1 = 0.25$ and Strain rate damping $c_2 = 0.25$

The dispersion relation plots of frequency ( $\omega$ ) versus propagation constant ( $\kappa$ ) are presented in Fig. 6(a, b, and c) for the aforementioned configurations, representing only viscous damping, only strain rate damping, and combined viscous and strain rate damping, respectively. Considering the time-dependent nature of frequency shift corresponding to propagation constants, the dispersion is visualized at multiple time instances ranging from 0 to 100 seconds for both softening and hardening systems. The evolution of these plots reveals a transition from undamped to linear system behavior over time.

Furthermore, the plots of frequency shift ( $\Delta\omega$ ) versus propagation constant ( $\kappa$ ) in Fig. 6(d-f) for all three damping configurations highlight a decreasing trend in frequency shift with increasing time. Notably, for systems with only strain rate damping ( $c_2$ ), the damping effect is proportional to  $\kappa^4$ , resulting in a more rapid decay for higher wavenumbers. Conversely, systems with viscous damping ( $c_1$ ) exhibit a damping effect independent of wavenumber. In damped systems, nonlinear effects tend to diminish with decaying amplitudes.

First, we investigate the influence of viscous damping on the behavior of a system under varying propagation constants. Specifically, we focus on three distinct propagation constants, namely  $\kappa = 0.50, 1.00$ , and  $1.50$ , examining the dynamics of frequency shift over time. This

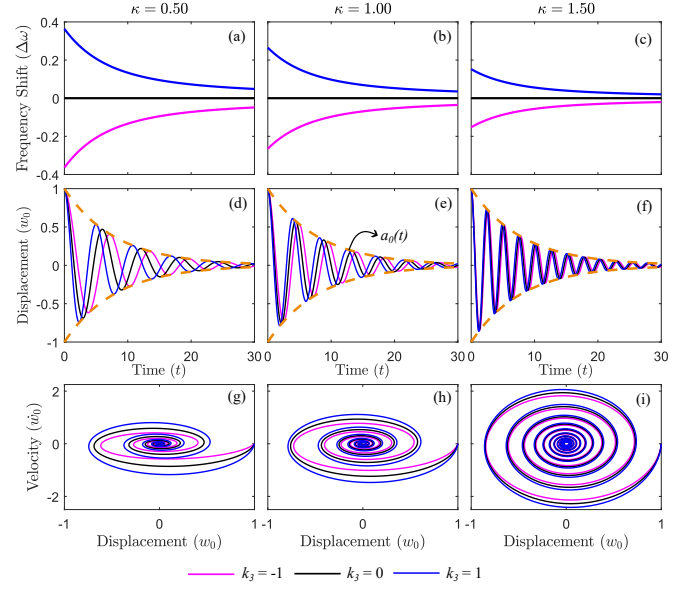


FIG. 7. Damped nonlinear system with only viscous damping  $c_1 = 0.25$ ;  $c_2 = 0$  and  $k_1 = 1$ . (a), (b) and (c) shows a frequency shift ( $\Delta\omega$ ) versus time ( $t$ ) propagation constant  $\kappa = 0.50, 1.00$  and  $1.50$  respectively; (d), (e) and (f) show displacement ( $w_0$ ) versus time ( $t$ ) at the middle node of beam for propagation constant  $\kappa = 0.50, 1.00$  and  $1.50$  respectively and yellow dashed line is for amplitude envelope; (g), (h) and (i) shows phase portraits for middle node of beam for propagation constant  $\kappa = 0.50, 1.00$  and  $1.50$  respectively.

analysis is presented through plots in Fig. 7 (a, b, and c), where differentiating characteristics of damping effects are delineated for hardening (magenta line), softening (blue line), and linear (black line) systems. Our findings reveal a convergence of frequency shifts to zero over time across all propagation constants, indicating consistent trends irrespective of the system's nonlinearity. Furthermore, the temporal evolution of displacement at the beam's middle node is depicted in Fig. 7 (d, e, and f) for  $\kappa = 0.50, 1.00$ , and  $1.50$ , respectively. In addition to these plots, the amplitude modulation, represented by the yellow dashed line, remains uniform across different propagation constants and types of nonlinearity. Additionally, phase portraits are presented in Fig. 7 (g, h, and i) for the same set of propagation constants. Notably, higher propagation constants correspond to higher velocities, indicative of increased frequencies. Moreover, the phase portraits elucidate distinct shifts from linear frequency: positive for hardening and negative for softening nonlinear systems, underscoring characteristic nonlinear behaviors. Additionally, we explore the impact of strain rate damping on systems characterized by diverse propagation constants. The damping factor corresponding to strain rate damping is proportional to  $\kappa^4$ , so the amplitude modulation varies with the propagation constant. The temporal evolution of frequency shift is depicted in Fig. 8(a, b, and c) for propagation constants  $\kappa = 0.5, 1.00$ , and  $1.50$ , respectively. Notably, the rate

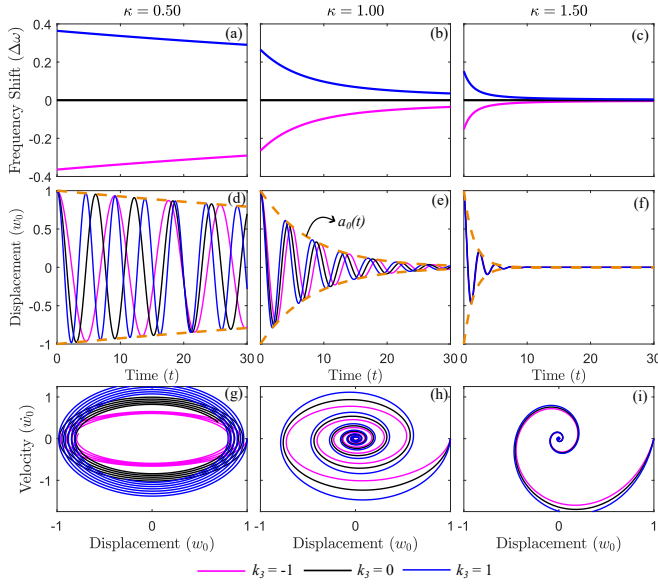


FIG. 8. Damped nonlinear system with only viscous damping  $c_1 = 0; c_2 = 0.25$  and  $k_1 = 1$ . (a), (b) and (c) shows a frequency shift ( $\Delta\omega$ ) versus time ( $t$ ) propagation constant  $\kappa = 0.50, 1.00$  and  $1.50$  respectively; (d), (e) and (f) show displacement ( $w_0$ ) versus time ( $t$ ) at the middle node of beam for propagation constant  $\kappa = 0.50, 1.00$  and  $1.50$  respectively and the yellow dashed line is for amplitude envelope; (g), (h) and (i) shows phase portraits for the middle node of beam for propagation constant  $\kappa = 0.50, 1.00$  and  $1.50$  respectively.

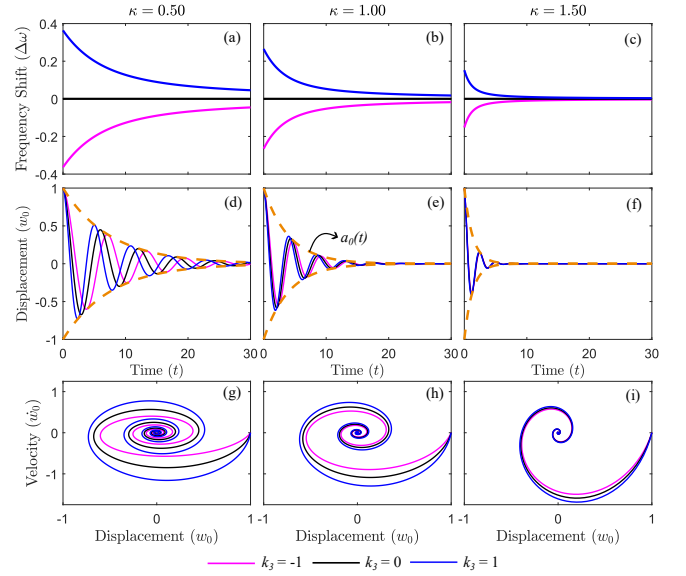


FIG. 9. Damped nonlinear system with only viscous and strain rate damping combine  $c_1 = 0.25; c_2 = 0.25$  and  $k_1 = 1$ . (a), (b) and (c) shows a frequency shift ( $\Delta\omega$ ) versus time ( $t$ ) propagation constant  $\kappa = 0.50, 1.00$  and  $1.50$  respectively; (d), (e) and (f) show displacement ( $w_0$ ) versus time ( $t$ ) at the middle node of beam for propagation constant  $\kappa = 0.50, 1.00$  and  $1.50$  respectively and yellow dashed line is for amplitude envelope; (g), (h) and (i) shows phase portraits for middle node of beam for propagation constant  $\kappa = 0.50, 1.00$  and  $1.50$  respectively.

of decrease in frequency shift varies significantly across different propagation constants, being notably slower for lower values of  $\kappa$  and more rapid for higher values. Upon examining the amplitude modulation for all three propagation constants in Fig. 8(d, e, and f), it becomes apparent that the decay progresses at a relatively slow pace when  $\kappa = 0.50$ , whereas it accelerates swiftly when  $\kappa = 1.50$ . Similarly, the phase portraits presented in Fig. 8(g, h, and i) for  $\kappa = 0.5, 1.00$ , and  $1.50$  reinforce these observations, providing further clarity on the relationship between propagation constants and damping effects.

Furthermore, we examine the combined effect of both types of damping, as illustrated in Fig. 9. The plots depicting frequency shift over time are presented in Fig. 9(a, b, and c) for propagation constants  $\kappa = 0.5, 1.00$ , and  $1.50$ , respectively. Notably, the frequency shift gradually diminishes over time, converging towards the frequency of a linear system as time progresses. Additionally, the amplitude modulation shown in Fig. 9(d, e, and f) elucidates the increase in decay rate with propagation constant. In the lower range of propagation constants, amplitude decay is primarily attributed to viscous damping, whereas in the higher range, it is predominantly due to strain rate damping. To gain a clearer understanding, phase portraits are depicted in Fig. 8(g, h, and i) for  $\kappa = 0.5, 1.00$ , and  $1.50$ , respectively, reinforcing these observations.

#### IV. SUMMARY AND CONCLUSION

In this paper, the analytical closed-form dispersion relation equation for a damped slender elastic beam periodically supported by cubic nonlinear springs has been derived. To derive the nonlinear dispersion relation, the method of multiple scales is employed after introducing a scaling parameter,  $\epsilon$ , to account for slow time scales. Through this approach, the governing partial differential equation is expanded and solved iteratively, leading to expressions for complex amplitudes and frequency shifts. The amplitude modulation and frequency shift equations are derived as functions of damping coefficients and amplitude of the initial plane wave. The proposed theory for nonlinear wave propagation was validated numerically through finite element formulation. Numerical simulations were carried out using the MATLAB ODE45 function. The results of numerical and analytical solutions of the undamped and damped systems show consistency which affirms the validity of the developed analytical closed-form solutions.

An extensive investigation into wave dispersion in nonlinear undamped systems was carried out using an analytical plane wave solution. The key findings are as follows:

- For a given propagation constant, there will be an

amplitude-dependent shift in frequency. The positive shift is observed for hardening springs and negative for softening springs.

- Since the frequency shift is inversely proportional to the frequency of the corresponding linear system, nonlinearity has a greater impact at low propagation constants or low frequencies and it reduces with increasing propagation constants or frequencies.

Additionally, the study thoroughly investigates the effects of viscous damping and strain rate damping, both individually and in combination. Through the analysis of amplitude modulation and phase portraits, distinct behaviors were elucidated across systems with different propagation constants and damping effects. The key findings are summarized as follows:

- Over time, the amplitude-dependent frequency shift decreases and eventually reaches zero, regardless of propagation constants or damping type. Hence, the influence of nonlinearity diminishes over time in a damped system.
- Analysis of viscous damping revealed a consistent decay in frequency shifts over time across all propagation constants, while amplitude modulation remained uniform.
- The rate at which the amplitude decays in a system with strain rate damping is directly proportional to the fourth power of propagation constant. This leads to a relatively sluggish convergence of the frequency shift to zero at low propagation constants, but quite more rapid convergence at higher propagation constants.
- Furthermore, the combined effect of both types of damping illustrated a gradual convergence of frequency towards that of a linear system as time progressed. Amplitude decay was primarily attributed to viscous damping at lower propagation constants and strain rate damping at higher propagation constants.

This study has the following major contributions:

- A closed-form equation for amplitude-dependent dispersion relation has been derived for continuous systems, providing a valuable analytical tool for further research in this domain.
- Furthermore, the incorporation of damping effects into the analysis of nonlinear dispersion offers insights into emulating real-world behavior, enhancing the applicability and relevance of the findings.

## ACKNOWLEDGEMENTS

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## Appendix A: Multiple spatial scales approach

The nonlinear partial differential equation, as shown in Eq. (4), involves the independent variables  $x$  and  $t$ , which correspond to the spatial and temporal dimensions, respectively. By employing a multiscale method, additional scales are introduced to account for long spatial scales, defined as  $X_1 = \epsilon x$ , which augment the original spatial scale  $X_0 = x$ . Consequently, the spatial derivatives can be expressed as

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X_0} + \epsilon \frac{\partial}{\partial X_1} \quad (\text{A1})$$

Further, the solution can be obtained as

$$w(x, t, \epsilon) = \sum_{n=0}^1 \epsilon^n w_n(X_0, X_1, t) + O(\epsilon) \quad (\text{A2})$$

By substituting the solution given in Eq. (A2) into the partial differential equation given in Eq. (4) and employing the spatial derivatives provided in Eq. (A1), the governing partial differential equation can be expressed through the consolidation of terms with similar powers of  $\epsilon$  as.

$$R_0 + R_1\epsilon + O(\epsilon^2) = 0 \quad (\text{A3})$$

where

$$\begin{aligned} R_0 : \frac{\partial^4 w_0}{\partial X_0^4} + \frac{\partial^2 w_0}{\partial t^2} + k_1 w_0 &= 0 \\ R_1 : \frac{\partial^4 w_1}{\partial X_0^4} + 4 \frac{\partial^4 w_0}{\partial X_1 \partial X_0^3} + \frac{\partial^2 w_1}{\partial t^2} + k_1 w_1 \\ &+ c_1 \frac{\partial w_0}{\partial t} + c_2 \frac{\partial^5 w_0}{\partial t \partial X_0^4} + k_3 w_0^3 = 0 \end{aligned} \quad (\text{A4})$$

The solutions  $w_0$ , and  $w_1$  can be obtained by solving equations  $R_0 = 0$  and  $R_1 = 0$  in progression.

In the case of flexural wave solution in infinite structure, the boundary complexities can be ignored and the plane wave solution can be assumed for the equation  $R_0 = 0$ .

$$w_0 = A(X_1) e^{i(\kappa X_0 - \omega t)} + \bar{A}(X_1) e^{-i(\kappa X_0 - \omega t)} \quad (\text{A6})$$

Substituting Eq. (A6) in Eq. (A4) the dispersion relation for the linear system can be derived as

$$\kappa^4 - \omega^2 + k_1 = 0 \quad (\text{A7})$$

Further, substituting Eq. (A6) in Eq. (A5), the following equation can be obtained

$$\begin{aligned} & \frac{\partial^2 w_1}{\partial t^2} + \frac{\partial^4 w_1}{\partial X_0^4} + k_1 w_1 \\ & = \left( 4i\kappa^3 \frac{\partial A}{\partial X_1} + i c_1 \omega A + i c_2 \kappa^4 \omega A - 3 k_3 A^2 \bar{A} \right) e^{i(\kappa X_0 - \omega t)} \\ & - k_3 A^3 e^{i(3\kappa X_0 - 3\omega t)} + cc \end{aligned} \quad (\text{A8})$$

The particular solution of Eq. (A8) contains secular terms which lead to nonuniform expansion in scaled time. Since the linear operator  $\left( \frac{\partial^2}{\partial t^2} + \frac{\partial^4}{\partial X_0^4} + k_1 \right)$  is self-adjoint, as demonstrated in the appendix A, the solvability condition for eliminating secular terms can be derived. This condition is obtained by equating the forcing terms responsible for generating the secular terms to zero, which leads to the following partial differential equation governing the amplitude  $A$ .

$$\begin{aligned} & 4i\kappa^3 \frac{\partial A}{\partial X_1} + i c_1 \omega A + i c_2 \kappa^4 \omega A - 3 k_3 A^2 \bar{A} = 0 \\ & i \frac{\partial A}{\partial X_1} = -iA \left( \frac{c_1 + c_2 \kappa^4}{4\kappa^3} \right) \omega + \frac{3}{4\kappa^3} k_3 A^2 \bar{A} \end{aligned} \quad (\text{A9})$$

The solution of  $A$  in polar form can be assumed as  $A = \frac{1}{2}a(X_1)e^{i\beta(X_1)}$ ,  $\bar{A} = \frac{1}{2}a(X_1)e^{-i\beta(X_1)}$  and substituting them into Eq. A9 the following equations can be obtained.

$$\begin{aligned} & i \frac{1}{2} \left( e^{i\beta} \frac{\partial a}{\partial X_1} + i e^{i\beta} a \frac{\partial \beta}{\partial X_1} \right) \\ & = -i \left( \frac{c_1 + c_2 \kappa^4}{4\kappa^3} \right) \omega \frac{a}{2} e^{i\beta} + \frac{3}{4\kappa^3} k_3 e^{i\beta} \frac{a^3}{8} \end{aligned} \quad (\text{A10})$$

Further separating the real and imaginary parts following equations can be obtained.

$$\frac{\partial a}{\partial X_1} = - \left( \frac{c_1 + c_2 \kappa^4}{4\kappa^3} \right) \omega a \quad (\text{A11})$$

$$\frac{\partial \beta}{\partial X_1} = - \frac{3}{16\kappa^3} k_3 a^2 \quad (\text{A12})$$

The amplitude modulation can be obtained by solving Eq. (A11) as

$$a = a_0 e^{- \left( \frac{c_1 + c_2 \kappa^4}{2\kappa^3} \right) \omega X_1} \quad (\text{A13})$$

Further, the wave number shift can be obtained by substituting Eq. (A13) into Eq. (A12) as follows.

$$\begin{aligned} & \frac{\partial \beta}{\partial X_1} = - \frac{3}{16\kappa^3} k_3 a_0^2 e^{- \left( \frac{c_1 + c_2 \kappa^4}{2\kappa^3} \right) \omega X_1} \\ & \beta = \frac{3}{8} \frac{k_3 a_0^2 \omega^{-1}}{(c_1 + c_2 \kappa^4)} e^{- \left( \frac{c_1 + c_2 \kappa^4}{2\kappa^3} \right) \omega X_1} + \beta_0 \end{aligned} \quad (\text{A14})$$

Further, the initial condition has been assumed as at  $X_1 = 0, \beta = 0$  and so,  $\beta_0 = -\frac{3}{8} \frac{k_3 a_0^2 \omega^{-1}}{(c_1 + c_2 \kappa^4)}$ . The shift in frequency can be determined as

$$\begin{aligned} \beta & = \frac{3}{8} \frac{k_3 a_0^2 \omega^{-1}}{(c_1 + c_2 \kappa^4)} e^{- \left( \frac{c_1 + c_2 \kappa^4}{2\kappa^3} \right) \omega X_1} - \frac{3}{8} \frac{k_3 a_0^2 \omega^{-1}}{(c_1 + c_2 \kappa^4)} \\ \beta & = - \frac{3}{8} \frac{k_3 \omega^{-1} a_0^2}{(c_1 + c_2 \kappa^4)} \left( 1 - e^{- \left( \frac{c_1 + c_2 \kappa^4}{2\kappa^3} \right) \omega X_1} \right) \\ \beta & = - \frac{3}{8} \frac{k_3 \omega^{-1} a_0^2}{(c_1 + c_2 \kappa^4)} \left( 1 - e^{- \left( \frac{c_1 + c_2 \kappa^4}{2\kappa^3} \right) \omega \epsilon x} \right) \end{aligned} \quad (\text{A15})$$

Using the Eq. (A15), the shift in wavenumber can be determined. However, to obtain the frequency shift the concept of group velocity in amplitude modulation equation has been used. Further, from differentiating the Eq. (A7) with propagation constant  $\kappa$  the group velocity ( $v_g = \frac{d\omega}{d\kappa}$ ) can be obtained as

$$\begin{aligned} & 4\kappa^3 - 2\omega \frac{d\omega}{d\kappa} = 0 \\ & v_g = \frac{d\omega}{d\kappa} = \frac{2\kappa^3}{\omega} \end{aligned} \quad (\text{A16})$$

Now, by substituting Eq. (A16) into Eq. (A13) and Eq. (A15), and incorporating  $v_g = x/t$  the following equation can be determined

$$a = a_0 e^{- (c_1 + c_2 \kappa^4) \epsilon t} \quad (\text{A17})$$

$$\beta = - \frac{3}{8} \frac{k_3 \omega^{-1} a_0^2}{(c_1 + c_2 \kappa^4)} \left( 1 - e^{- (c_1 + c_2 \kappa^4) \epsilon t} \right) \quad (\text{A18})$$

Equations A17 and A18 match exactly with the solutions obtained using temporal multiscales in Eqs. 19 and 21. Thus, it can concluded that both temporal and spatial methods of multiple scales can be used to determine nonlinear frequency shifts.

## Appendix B: Proof of self ad-joint operator

To prove the left-hand side of Eq.  $\left( \frac{\partial^2}{\partial T_0^2} + \frac{\partial^4}{\partial x^4} + k_1 \right)$  self adjointed, first the variable separation method has been used to get two separate ordinary differential equations (ODE) as follows. Let, the solution  $w(x, T_0) = f(x)g(T_0)$

$$\begin{aligned} & \left( \frac{\partial^2}{\partial T_0^2} + \frac{\partial^4}{\partial x^4} + k_1 \right) w(x, T_0) = 0 \\ & \left( \frac{\partial^2}{\partial T_0^2} + \frac{\partial^4}{\partial x^4} + k_1 \right) f(x)g(T_0) = 0 \\ & f(x) \frac{d^2 g(T_0)}{dT_0^2} + G(T_0) \frac{d^4 f(x)}{dx^4} + k_1 f(x)g(T_0) = 0 \end{aligned} \quad (\text{B1})$$



dividing whole Eq. (B1) with  $f(x)g(T_0)$  as

$$\frac{1}{g(T_0)} \frac{d^2 g(T_0)}{dT_0^2} + \frac{1}{f(x)} \frac{d^4 f(x)}{dx^4} + k_1 = 0 \quad (\text{B2})$$

Further,

$$\frac{1}{g(T_0)} \frac{d^2 g(T_0)}{dT_0^2} = -\frac{1}{f(x)} \frac{d^4 f(x)}{dx^4} - k_1 = \lambda \quad (\text{B3})$$

following this two separate ODEs can be written as

$$\frac{d^2 g(T_0)}{dT_0^2} - \lambda g(T_0) = 0 \quad (\text{B4})$$

$$\frac{d^4 f(x)}{dx^4} + (k_1 + \lambda) f(x) = 0 \quad (\text{B5})$$

The eigenvalue problem to solve is Eq. (B5) and the Linear operator in is  $(L = \frac{d^4}{dx^4})$ . Operator  $L$  can be called self ad-joint if inner products  $\langle Lu, v \rangle = \langle u, Lv \rangle$ .

$$\begin{aligned} \langle Lu, v \rangle &= \int_a^b \left( \frac{d^4 u(x)}{dx^4} \right) v(x) dx \\ &= \left[ \frac{d^3 u(x)}{dx^3} v(x) \right]_a^b - \int_a^b \frac{d^3 u(x)}{dx^3} \frac{dv(x)}{dx} dx \end{aligned} \quad (\text{B6})$$

$$\begin{aligned} &= \left[ \frac{d^3 u(x)}{dx^3} v(x) \right]_a^b + \left[ \frac{d^2 u(x)}{dx^2} \frac{dv(x)}{dx} \right]_a^b \\ &- \int_a^b \frac{d^2 u(x)}{dx^2} \frac{d^2 v(x)}{dx^2} dx \end{aligned} \quad (\text{B7})$$

Further,

$$\langle u, Lv \rangle = \int_a^b u(x) \left( \frac{d^4 v(x)}{dx^4} \right) dx \quad (\text{B8})$$

$$= \left[ \frac{d^3 v(x)}{dx^3} u(x) \right]_a^b - \int_a^b \frac{d^3 v(x)}{dx^3} \frac{du(x)}{dx} dx \quad (\text{B9})$$

$$\begin{aligned} &= \left[ \frac{d^3 v(x)}{dx^3} u(x) \right]_a^b + \left[ \frac{d^2 v(x)}{dx^2} \frac{du(x)}{dx} \right]_a^b \\ &- \int_a^b \frac{d^2 v(x)}{dx^2} \frac{d^2 u(x)}{dx^2} dx \end{aligned} \quad (\text{B10})$$

After applying appropriate boundary conditions the Eq. (B7) and Eq. (B10) can be proved to be the same. For brevity in the case of the simply supported beam, the boundary conditions  $u(a) = 0, v(a) = 0, u(b) = 0$ , and  $v(b) = 0$  can be substituted in the Eq. (B7) and Eq. (B10). The inner product is:

$$\langle Lu, v \rangle = \langle u, Lv \rangle = - \int_a^b \frac{d^2 u(x)}{dx^2} \frac{d^2 v(x)}{dx^2} dx \quad (\text{B11})$$

Hence, It can be said that the operator  $\left( \frac{\partial^2}{\partial T_0^2} + \frac{\partial^4}{\partial x^4} + k_1 \right)$  is self adjointed.

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